

- Today :
- Explicit stable reduction
 - Uniqueness of stable limit
 - Gluing morphisms + boundary divisors
 - Forgetful morphisms
-

Explicit stable reduction

Recall Theorem. R DVR w/ $K = \text{Frac}(R)$. $\Delta = \text{Spec}(R)$, $\Delta^* = \text{Spec}(K)$.

$(C^* \rightarrow \Delta^*, s_1^*, \dots, s_n^*)$ n -pointed stable curve of genus g .

$\Rightarrow \exists$ finite cover $\Delta' \rightarrow \Delta$ of DVRs

& $(C' \rightarrow \Delta', s_1', \dots, s_n')$ family of n -pointed stable curves of genus g that extends $C^* \times_{\Delta^*} \Delta'^* \rightarrow \Delta'^*$.

Construction : Step 0 reduce to case when $C^* \rightarrow \Delta^*$ smooth

Step 1 find a flat extension $C \rightarrow \Delta$

Step 2 resolution of singularities $\tilde{C} \rightarrow C$ s.t. $(\tilde{C}_0)_{\text{red}}$ nodal

$\text{Spec } R \rightarrow \text{Spec } R$
 $(t \mapsto t^n)$

when $R = k[[t]]_{(t)}$

Step 3 take deg m ramified base extension $\Delta' \rightarrow \Delta$ so that $(C \times_{\Delta} \Delta')_0^{\text{nor}}$ reduced and nodal (assuming char 0)

Step 4 take minimal model $\tilde{C}_{\min} \rightarrow C$ then stabilize (contract all rational tails and bridges)

In explicit computation, the biggest challenge is the computation of the $(C \times_{\Delta} \Delta')_0^{\text{nor}}$ in Step 3.

Strategy Break the extension to prime degree ones!

Prop. $C \rightarrow \Delta$ generically smooth, proper, flat family s.t. $(C_0)_{\text{red}}$ nodal.

Let $C_0 = \sum a_i D_i$, $\Delta' \rightarrow \Delta$ degree p (prime) extension,

$$C' = C \times_{\Delta} \Delta', \quad \tilde{C}' = C'^{\text{nor}}.$$

Then $\tilde{C}' \rightarrow C$ is a branched cover ramified over $\sum_{i \neq 0} D_i$.

Ex. (stable reduction of A_{2k+1} -singularity) (char 0)

$C \rightarrow \mathbb{A}_t^1$ where $C: y^2 = x^{2k+1} + t$. Q. Stable limit as $t \rightarrow 0$?

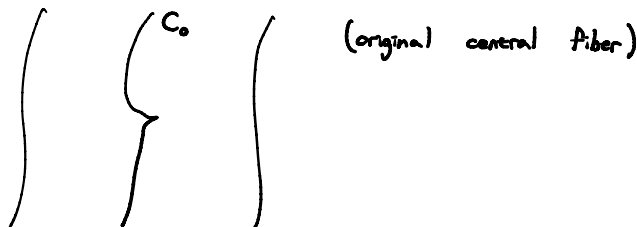
Already have smooth generic fiber & a flat limit $\Rightarrow$ start at Step 2.

Step 2 Find resolution of singularities $\tilde{C} \rightarrow C$ s.t. $(\tilde{C}_0)_{\text{red}}$ nodal.

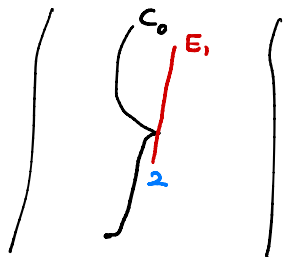
We will repeatedly blow-up the (reduced) singular point in central fiber.

Notation

$\left\{ \begin{array}{l} \text{Old coordinates : } x, y \\ \text{New coordinates : } \tilde{x}, \tilde{y} \end{array} \right.$	$\left\{ \begin{array}{l} \text{Chart \#1 : } \tilde{x} = x, \tilde{y} = y/x \\ \text{Chart \#2 : } \tilde{x} = x/y, \tilde{y} = y \end{array} \right.$
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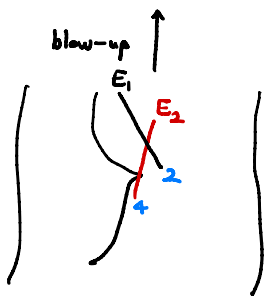
blow-up $\uparrow$



Blow-up #1

$$\text{Chart \#1: } y^2 - x^{2k+1} = \tilde{x}^2 (\tilde{y}^2 - \tilde{x}^{2k-1})$$

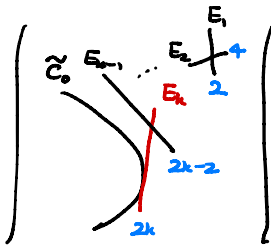
$\underbrace{\hspace{1.5cm}}_{E_1}$



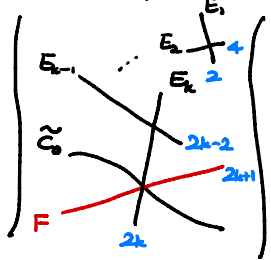
blow-up ↑

...

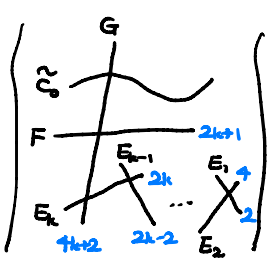
blow-up ↑



blow-up ↑



blow-up ↑



Blow-up #2

$$\text{Chart \#1: } x^2(y^2 - x^{2k-1}) = \underbrace{\tilde{x}^4}_{E_2} (\underbrace{\tilde{y}^2}_{E_2} - \underbrace{\tilde{x}^{2k-1}}_{E_2})$$

$$\text{Chart \#2: } x^2(y^2 - x^{2k-1}) = \underbrace{\tilde{x}^2}_{E_1} \underbrace{\tilde{y}^4}_{E_2} (1 - \underbrace{\tilde{x}^{2k-1}}_{E_1} \underbrace{\tilde{y}^{2k-3}}_{E_2})$$

Blow-up #k

$$\text{Chart \#1: } x^{2k-2}(y^2 - x^2) = \underbrace{\tilde{x}^{2k}}_{E_k} (\underbrace{\tilde{y}^2}_{\tilde{C}_0} - \underbrace{\tilde{x}}_{\tilde{C}_0})$$

normalization of the original central fiber

Blow-up #k+1

$$\text{Chart \#2: } x^{2k}(y^2 - x) = \underbrace{\tilde{x}^{2k}}_{E_k} \underbrace{\tilde{y}^{2k+1}}_F (\underbrace{\tilde{y} - \tilde{x}}_{\tilde{C}_0})$$

Blow-up #k+2

$$\text{Chart \#1: } x^{2k}y^{2k+1}(y - x) = \underbrace{\tilde{x}^{4k+2}}_G \underbrace{\tilde{y}^{2k+1}}_F (\underbrace{\tilde{y} - 1}_{\tilde{C}_0})$$

↪ (central fiber) = $\tilde{C}_0 + (4k+2)G + (2k+1)F + \sum_{i=1}^k 2iE_i$

Step 3 take $\Delta' \rightarrow \Delta$ s.t. $(C \times_{\Delta} \Delta')^{\text{nor}}_0$ is reduced + nodal

(for simplicity, assume $2k+1$ prime)

By Prop., $C' \xrightarrow{\pi} C$ defined by the base change $t \mapsto t^{2k+1}$

is ramified over $\tilde{C}_0 + \sum E_i$.

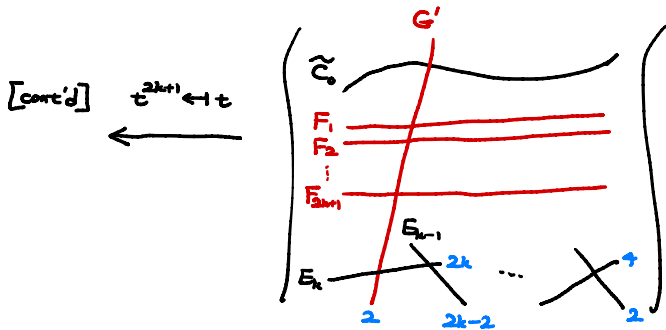
$$\left\{ \begin{array}{l} \pi^{-1}(F) \xrightarrow{2k+1:1} F \text{ unramified. } \Rightarrow \pi^{-1}(F) = F_1 \cup \dots \cup F_{2k+1} \text{ } (F_i: \text{sm rat'l curves}) \\ G' = \pi^{-1}(G) \xrightarrow{2k+1:1} G \cong \mathbb{P}^1 \text{ ramified over 2 pts} \end{array} \right.$$

with ramification indexes $2k+1$.

$$\begin{aligned} \text{RH} \quad \left\{ \begin{array}{l} 2g(G') - 2 = (2k+1)(g(\mathbb{P}^1) - 2) + \deg R = -2 \\ \Rightarrow g(G') = 0. \end{array} \right. \end{aligned}$$

$$\begin{aligned} \text{Over } \Delta, \text{ (central fiber)} &= (2k+1) \tilde{C}_0 + (4k+2) G' \\ &\quad + (2k+1) \sum F_i + (2k+1) \sum 2i E_i \end{aligned}$$

$$\text{Over } \Delta', \text{ (central fiber)} = \tilde{C}_0 + 2G' + \sum F_i + \sum 2i E_i$$



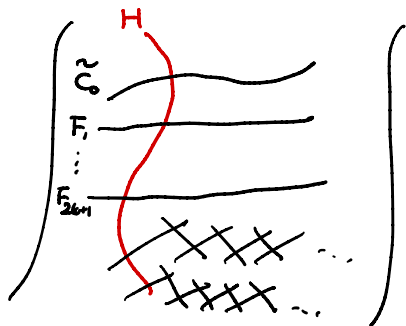
Next, base change by $t \mapsto t^2$ and normalize.

New surface is a $2:1$ cover ramified over $\tilde{C}_0 + \sum F_i$.

$$H = \pi^{-1}(G') \xrightarrow{2:1} G' \cong \mathbb{P}^1 \text{ ramified over } 2k+2 \text{ pts.}$$

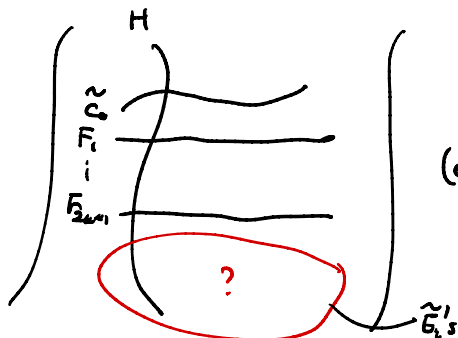
$\Rightarrow H$ is a hyperelliptic curve of genus k .

[cont'd] $t^2 \leftarrow t$



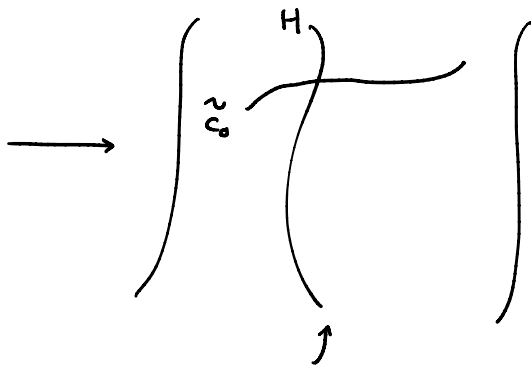
(everything is reduced except $\tilde{E}_i$'s)

$t^k \leftarrow t$
 $\vdots$
 $t^2 \leftarrow t$
 $\leftarrow$



(everything is reduced)

Step 4 Stabilize (contract rational tails)



stable limit = nodal union of $\tilde{C}_0$ (nor. of original central fiber) and some hyperelliptic curve w/ genus k H

More details and info: [HM] Moduli of curves, Chapter 3.C

Uniqueness of stable limit

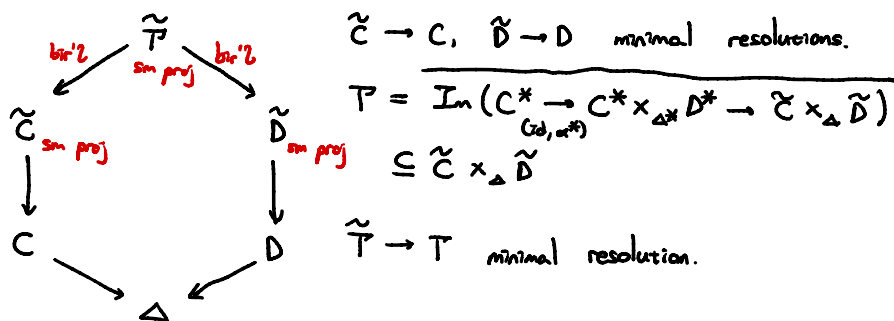
Prop. R DVR w/ $K = \text{Frac}(R)$. $\Delta = \text{Spec}(R)$, $\Delta^* = \text{Spec}(K)$.

$(C \rightarrow \Delta, \sigma_1, \dots, \sigma_n)$, $(D \rightarrow \Delta, \tau_1, \dots, \tau_n)$ families of n -pointed stable curves.

$\alpha^* : C^* \rightarrow D^*$ isomorphism over Δ^* w/ $\tau_i^* = \alpha^* \circ \sigma_i^*$.

$\Rightarrow \exists! \alpha : C \xrightarrow{\sim} D$ that extends α^* & $\tau_i = \alpha \circ \sigma_i$.

pf. (for simplicity, assume $n = 0$)



$$\Rightarrow T(\tilde{C}, w_{\tilde{C}/\Delta}^{\otimes k}) \cong T(\tilde{T}, w_{\tilde{T}/\Delta}^{\otimes k}) \cong T(\tilde{D}, w_{\tilde{D}/\Delta}^{\otimes k}) \quad (\forall k \geq 0)$$

(cf. Hartshorne II. 8.19)

Since C, D are stable models of $\tilde{C}, \tilde{D}$,

$$C \cong \text{Proj} \bigoplus_{k \geq 0} T(\tilde{C}, w_{\tilde{C}/\Delta}^{\otimes k})$$

$$D \cong \text{Proj} \bigoplus_{k \geq 0} T(\tilde{D}, w_{\tilde{D}/\Delta}^{\otimes k})$$

■

Upshot • $\overline{\mathcal{M}}_{g,n}$ is proper

• By Keel - Mori, $\exists$ coarse moduli space $\exists \overline{\mathcal{M}}_{g,n} \rightarrow \mathcal{M}_{g,n}$
 $\uparrow$
 proper alg. space

Gluing morphisms

Prop. $\exists$ canonical finite morphisms of alg. stacks

$$(a) \overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{g-1,m} \rightarrow \overline{\mathcal{M}}_{g,n+m-2}$$

$$\left(\begin{array}{c} \text{Diagram 1: Blue curve with point } p_n \\ \text{Diagram 2: Green curve with point } p'_m \end{array} \right) \mapsto \text{Diagram 3: Glued curve with points } p_n, p'_m$$

$$(C, p_1, \dots, p_n) \quad (C', p'_1, \dots, p'_m) \quad (C \cup C' / p_n \sim p'_m, p_1, \dots, p_{n-1}, p'_1, \dots, p'_{m-1})$$

$$(b) \overline{\mathcal{M}}_{g-1,n} \rightarrow \overline{\mathcal{M}}_{g,n-2}$$

$$\begin{array}{c} \text{Diagram 1: Green curve with points } p_{n-1}, p_n \\ \text{Diagram 2: Glued curve with points } p_1, \dots, p_{n-2} \end{array} \mapsto$$

$$(C, p_1, \dots, p_n) \quad (C / p_{n-1} \sim p_n, p_1, \dots, p_{n-2})$$

pf. (for simplicity, assume (a) $n = m = 1$ (b) $n = 2$)

Idea: use pushouts!

(a) $(\tau \xrightarrow{\pi} S, \sigma), (\tau' \xrightarrow{\pi'} S', \sigma')$ two families of 1-pointed stable curves.

$$\begin{array}{ccc} S & \xrightarrow{\sigma} & \mathcal{C} \\ \sigma' \downarrow & & \downarrow \\ \mathcal{C}' & \longrightarrow & \tilde{\mathcal{C}} \end{array}$$

Need to show $\tilde{\mathcal{C}} \rightarrow S$ is a nodal family of curves s.t. all geometric fibers are stable.

stable locally

$$\forall s \in S: \begin{cases} \sigma(s) \text{ smooth point in } \mathcal{C} \\ \sigma'(s) \text{ " " } \mathcal{C}' \end{cases}$$

$$\begin{array}{ccc} \text{Spec } A & \longrightarrow & \text{Spec } A[y] \\ \downarrow & & \downarrow \\ \text{Spec } A[x] & \longrightarrow & \text{Spec } A[x] \times_A A[y] \cong \text{Spec } A[x, y] / (xy). \end{array}$$

$\Rightarrow \mathcal{C} \rightarrow S$ is étale locally $\text{Spec } A[x, y] / (xy) \rightarrow \text{Spec } A$
and hence a nodal family of curves.

Lastly, $\tilde{\mathcal{C}}_s = \text{nodal union of } \mathcal{C}_s \text{ \& } \mathcal{C}'_s \text{ which is stable.}$

(b) $(\mathcal{C} \rightarrow S, \sigma_1, \sigma_2)$ 2-prod family of stable curves.

$$\begin{array}{ccc} \text{SL}_2 & \xrightarrow{\sigma_1 \cup \sigma_2} & \mathcal{C} \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathcal{C} \end{array}$$

$\Rightarrow$ Similarly, $\mathcal{C} \rightarrow S$ is étale locally

$$\text{Spec } A[x, y] / (y^2 - x^2(x+1)) \rightarrow \text{Spec } A.$$

□

Boundary divisors of $\overline{\mathcal{M}}_g$

$$\delta_0 = \text{Im}(\overline{\mathcal{M}}_{g-1,2} \rightarrow \overline{\mathcal{M}}_g)$$

$$\delta_i = \text{Im}(\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \rightarrow \overline{\mathcal{M}}_g) \left(1 \leq i \leq \left\lfloor \frac{g}{2} \right\rfloor \right)$$

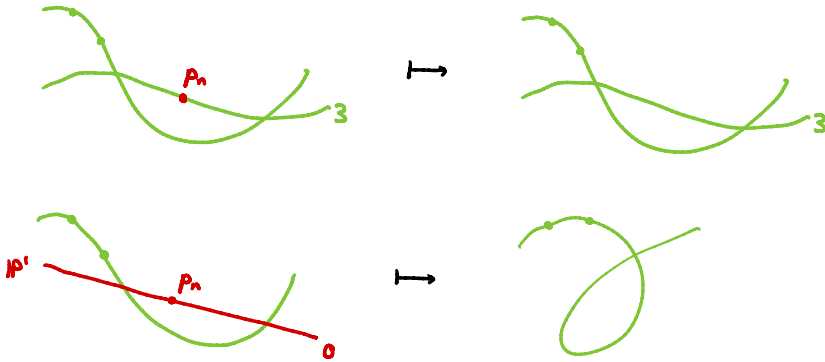
$\Rightarrow$ indeed divisors!

Fact $\delta = \delta_0 \cup \dots \cup \delta_{\lfloor \frac{g}{2} \rfloor}$ is normal crossing.

Forgetful morphisms

Prop. $\exists$ canonical morphism of alg. stacks $\overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n-1}$

$$\begin{aligned} ((C, p_1, \dots, p_n) &\mapsto (C^{st}, p_1, \dots, p_{n-1})) \\ &\uparrow \\ &\text{stable model of } (C, p_1, \dots, p_{n-1}) \end{aligned}$$



Def. (universal family $U_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$)

Objects of $U_{g,n}$ over $S =$ families of n -pointed stable curves

$(\mathcal{C} \rightarrow S, \sigma_1, \dots, \sigma_n)$ w/ section

$\sigma_{n+1} : S \rightarrow \mathcal{C}$. (no condition on σ_{n+1})

(Universal property?) $\forall (\mathcal{C} \rightarrow S, \sigma_1, \dots, \sigma_n)$ family of n -pointed stable curves:

$$\begin{array}{ccc} \mathcal{C} & \dashrightarrow & U_{g,n} \\ \downarrow & \square & \downarrow \\ S & \dashrightarrow & \overline{\mathcal{M}}_{g,n} \end{array}$$

$$\exists \overline{\mathcal{M}}_{g,n+1} \rightarrow U_{g,n} \left((\mathcal{C} \rightarrow S, \sigma_1, \dots, \sigma_n, \sigma_{n+1}) \mapsto ((\mathcal{C}^{st} \rightarrow S, \sigma_1^{st}, \dots, \sigma_n^{st}), \sigma_{n+1}^{st}) \right)$$

where $\pi : \mathcal{C} \rightarrow \mathcal{C}^{st}$, $\sigma_i^{st} = \pi \circ \sigma_i$.

Easy to check $\overline{\mathcal{M}}_{g,n+1} \rightarrow U_{g,n}$
 $\downarrow \hookrightarrow \downarrow$
 $\overline{\mathcal{M}}_{g,n}$

Fact. $\overline{\mathcal{M}}_{g,n+1} \rightarrow U_{g,n}$ is an iso.

(Knudsen - Projectivity II)